Iterated Brownian Motion in Parabola-Shaped Domains

Erkan Nane*
Department of Mathematics
Purdue University
West Lafayette, IN 47906
enane@math.purdue.edu

February 1, 2008

Abstract

Iterated Brownian motion Z_t serves as a physical model for diffusions in a crack. If $\tau_D(Z)$ is the first exit time of this processes from a domain $D \subset \mathbb{R}^n$, started at $z \in D$, then $P_z[\tau_D(Z) > t]$ is the distribution of the lifetime of the process in D. In this paper we determine the large time asymptotics of $P_z[\tau_{P_\alpha}(Z) > t]$ which gives exponential integrability of $\tau_{P_\alpha}(Z)$ for parabola-shaped domains of the form $P_\alpha = \{(x,Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$, for $0 < \alpha < 1$, $A > 0$. We also obtain similar results for twisted domains in $\mathbb{R}^2$ as defined in [9]. In particular, for a planar iterated Brownian motion in a parabola $\mathcal{P} = \{(x,y) : \ x > 0, \ |y| < \sqrt{x}\}$ we find that for $z \in \mathcal{P}$

$$\lim_{t\to\infty} t^{-\frac{1}{7}} \log P_z[\tau_{\mathcal{P}}(Z) > t] = -\frac{7\pi^2}{2^{25/7}}.$$

Mathematics Subject Classification (2000): 60J65, 60K99.
Key words: Iterated Brownian motion, exit time, parabola-shaped domain.

*Supported in part by NSF Grant # 9700585-DMS

1 Introduction

Iterated Brownian motion has been of considerable interest to several authors in recent years; see for example Burdzy [4, 5], DeBlassie [8], Koshnevisan and Lewis [15] and references in these articles. Although this processes is not a Markov process (it does not satisfy the Chapman–Kolmogorov equations), it does have connections with the parabolic operator $\frac{1}{8}\Delta^2 - \frac{\partial}{\partial t}$, as described in Funaki [12] and DeBlassie [8].

In analogy with ordinary Brownian motion and diffusions, if $\tau_D(Z)$ is the first exit time of iterated Brownian motion from domain D, started at $z \in D$, $P_z[\tau_D(Z) > t]$ provides a measure of the lifetime of the process in D. The tail distribution of $\tau_D(Z)$ has a double integral representation in terms of the probability density function of the Brownian motion, as given in [8]. This representation can then be used to compute the asymptotics of the tail distribution of $\tau_D(Z)$ from which one can then obtain the sharp order of integrability of $\tau_D(Z)$. The goal of this paper is to do exactly this when the domain is a general parabola in $\mathbb{R}^n$, or a twisted domain in $\mathbb{R}^2$.

To define the iterated Brownian motion Z_t started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. Two-sided Brownian motion is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then the iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X(Y_t), \quad t \geq 0.$$

In $\mathbb{R}^n$, one requires $X^\pm$ to be independent $n-$dimensional Brownian motions. This is the version of the iterated Brownian motion due to Burdzy. Our choice of independent one-dimensional Brownian motions is motivated by the pde connection to $\frac{1}{8}\Delta^2 - \frac{\partial}{\partial t}$ by Funaki [12] and DeBlassie [8] and by the interpretation of the process as a diffusion in a crack by Burdzy and Khoshnevisan [6].

The path properties of this process have been studied by Burdzy in [4], and [5]. His works in particular imply the LIL(Law of Iterated Logarithm) for the iterated Brownian motion. Khoshnevisan and Lewis extend results in [5], to develop a stochastic calculus for iterated Brownian motion. The local time for the iterated Brownian motion has been studied by Csàki, Csörgő, Földes and Révész [7], Eisenbum and Shi [10] and Xiao [17].

In [8], DeBlassie obtains large time asymptotics of the tail distribution of the exit time of Z_t in bounded domains which have regular boundary and for general cones in $\mathbb{R}^n$. Let D be a domain in $\mathbb{R}^n$. Let

$$\tau_D(Z) = \inf\{t \geq 0 : \ Z_t \notin D\}$$

be the first exit time of Z_t from D. In this paper we prove the large time asymptotics of the tail distribution of $\tau_D(Z)$ when the domain D is a general parabola in $\mathbb{R}^n$, or a twisted domain in $\mathbb{R}^2$. These type of questions are now well understood in the case of Brownian motion [1], [2], [9], [16] and serve, together with DeBlassie's results in [8], as motivation for our results. Indeed, Bañuelos, DeBlassie and Smits showed in [1] that if $\tau_{\mathcal{P}}$ is the first exit time of the Brownian motion from the parabola $\mathcal{P} = \{(x,y) : x > 0, |y| < A\sqrt{x}\}$, $A > 0$, then there exist positive constants A_1 and A_2 such that for $z \in \mathcal{P}$

$$\begin{aligned} -A_1 \quad &\leq \quad \liminf_{t\to\infty} \ t^{-\frac{1}{3}} \ \log \ P_z[\tau_{\mathcal{P}} > t] \\ &\leq \quad \limsup_{t\to\infty} \ t^{-\frac{1}{3}} \ \log \ P_z[\tau_{\mathcal{P}} > t] \leq \ -A_2. \end{aligned}$$

More recently, Lifshits and Shi [16] found that the above limit exists for parabolic regions in any dimension. Let $0 < \alpha < 1$ and $A > 0$. We define the parabola-shaped domains as $P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$. Let τ_α be the first exit time of Brownian motion from P_α. Lifshits and Shi [16] proved that

$$\lim_{t\to\infty} t^{-(\frac{1-\alpha}{1+\alpha})} \log P_z[\tau_\alpha > t] = -l, \tag{1.1}$$

where

$$l = (\frac{1+\alpha}{\alpha}) \left(\frac{\pi \jmath_{(n-3)/2}^{2/\alpha}}{A^2 2^{(3\alpha+1)/\alpha}((1-\alpha)/\alpha)^{(1-\alpha)/\alpha}} \frac{\Gamma^2(\frac{1-\alpha}{2\alpha})}{\Gamma^2(\frac{1}{2\alpha})} \right)^{\frac{\alpha}{(\alpha+1)}}. \tag{1.2}$$

Here $\jmath_{(n-3)/2}$ denotes the smallest positive zero of the Bessel function $J_{(n-3)/2}$ and Γ is the Gamma function. In particular, since $\jmath_{-1/2} = \pi/2$, this limit is $3\pi^2/8$ for the parabola $\mathcal{P}$, for which $A = 1$, $n = 2$ and $\alpha = 1/2$.

The following is the first main result in this paper. For simplicity of notation we will use $\tau_\alpha(Z)$ instead of $\tau_{P_\alpha}(Z)$ to denote the first exit time of the process Z_t from P_α.

Theorem 1.1. *Let $0 < \alpha < 1$ and let $P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$. Then for $z \in P_\alpha$,*

$$\lim_{t\to\infty} t^{-(\frac{1-\alpha}{3+\alpha})} \log P_z[\tau_\alpha(Z) > t] = -(\frac{3+\alpha}{2+2\alpha})(\frac{1+\alpha}{1-\alpha})^{(\frac{1-\alpha}{3+\alpha})}\pi^{(\frac{2-2\alpha}{3+\alpha})}l^{(\frac{2+2\alpha}{3+\alpha})},$$

where l is the limit given by (1.2).

In the case of two dimensions and the parabola $\mathcal{P}$, $A = 1$ we obtain by substituting $l = 3\pi^2/8$ in Theorem 1.1

$$\lim_{t\to\infty} t^{-\frac{1}{7}} \log \ P_z[\tau_{\mathcal{P}}(Z) > t] = -\frac{7\pi^2}{2^{25/7}}.$$

Theorem 1.1 gives the sharp order of integrability for iterated Brownian motion in these regions.

Corollary 1.1. *Let P_α be a general parabolic domain as in Theorem 1.1. Then for $z \in P_\alpha$,*

$$E_z\left[\exp(b(\tau_\alpha(Z))^{\frac{1-\alpha}{\alpha+3}})\right]$$

is finite if

$$b < (\frac{3+\alpha}{2+2\alpha})(\frac{1+\alpha}{1-\alpha})^{(\frac{1-\alpha}{3+\alpha})}\pi^{(\frac{2-2\alpha}{3+\alpha})}l^{(\frac{2+2\alpha}{3+\alpha})},$$

and it is infinite if

$$b > (\frac{3+\alpha}{2+2\alpha})(\frac{1+\alpha}{1-\alpha})^{(\frac{1-\alpha}{3+\alpha})}\pi^{(\frac{2-2\alpha}{3+\alpha})}l^{(\frac{2+2\alpha}{3+\alpha})}.$$

Note that we do not know what happens in the corollary when

$$b = (\frac{3+\alpha}{2+2\alpha})(\frac{1+\alpha}{1-\alpha})^{(\frac{1-\alpha}{3+\alpha})}\pi^{(\frac{2-2\alpha}{3+\alpha})}l^{(\frac{2+2\alpha}{3+\alpha})}.$$

The proof of Corollary 1.1 follows from Theorem 1.1 and the fact that

$$E_z\left[\exp(b(\tau_\alpha(Z))^{\frac{1-\alpha}{\alpha+3}})\right] = \int_0^\infty (\frac{d}{dt}\exp(bt^{\frac{1-\alpha}{\alpha+3}}))P_z[\tau_\alpha(Z) > t]dt.$$

In [9], DeBlassie and Smits studied the tail behavior of the first exit time of the Brownian motion from twisted domains in the plane. Let $D \subset \mathbb{R}^2$ be a domain whose boundary consists of three curves (in polar coordinates)

$$\begin{aligned} &C_1: \quad \theta = f_1(r), \quad r \geq r_1 \\ &C_2: \quad \theta = f_2(r), \quad r \geq r_1 \\ &C_3: \quad r = r_1, \qquad f_2(r) \leq \theta \leq f_1(r) \end{aligned}$$

where f_1 and f_2 are smooth and the curves C_1 and C_2 do not cross:

$$0 < f_1(r) - f_2(r) < \pi, \quad r \geq r_1.$$

DeBlassie and Smits call D a twisted domain if there are constants $r_0 > 0$, $\gamma > 0$ and $p \in (0,1]$ and a smooth function $f(r)$ such that the curves $f_1(r)$ and $f_2(r)$, $r \geq r_0$, are obtained from $f(r)$ by moving out $\pm\gamma r^p$ units along the normal to the curve $\theta = f(r)$ at the point whose polar coordinates are $(r, f(r))$. They call γr^p the growth radius and $\theta = f(r)$ the generating curve. DeBlassie and Smits [9, Theorem1.1] have the following tail behavior of the first exit time of Brownian motion from twisted domains $D \subset \mathbb{R}^2$ with growth radius γr^p, $\gamma > 0$, $0 < p < 1$

$$\lim_{t\to\infty} t^{-(\frac{1-p}{1+p})} \log P_z[\tau_D > t] = -l_1 = -\left[\frac{\pi^{2p-1}}{\gamma 2^{2p}(1-p)^{2p}}\right]^{\frac{2}{p+1}} C_p \tag{1.3}$$

where

$$C_p = (1+p)\left[\frac{\pi^{2+p}}{8^p p^{2p}(1-p)^{1-p}} \frac{\Gamma^{2p}\left(\frac{1-p}{2p}\right)}{\Gamma^{2p}\left(\frac{1}{2p}\right)}\right]^{\frac{1}{p+1}}.$$

For these domains we have the following theorem which is the last main result in this paper.

Theorem 1.2. *Let $D \subset \mathbb{R}^2$ be a twisted domain with growth radius γr^p, $\gamma > 0$, $0 < p < 1$. Then*

$$\lim_{t\to\infty} t^{-(\frac{1-p}{p+3})} \log P_z[\tau_D(Z) > t] = -(\frac{3+p}{2+2p})(\frac{1+p}{1-p})^{(\frac{1-p}{3+p})} \pi^{(\frac{2-2p}{3+p})} l_1^{(\frac{2+2p}{3+p})},$$

where l_1 is the limit given by (1.3).

Notice the similarity of the limits in Theorems 1.1 and 1.2 except the constants l and l_1.

To obtain our results we follow the general ideas of DeBlassie [8] but with some key modifications. We use the asymptotics of $\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]$, where $\eta_{(-u,v)}$ is the first exit time of one-dimensional Brownian motion from the interval $(-u, v)$ and integration by parts to get away from assuming the asymptotics of the density of the exit times.

The paper is organized as follows. In §2 we give some preliminaries needed in the proofs of Theorems 1.1 and 1.2. In §3, we prove Theorems 1.1 and 1.2. In §4 we derive several technical results that are used in the asymptotics of $\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]$.

2 Preliminaries

In this section we prove some results which we will use in section 3. In what follows we will write $f \approx g$ and $f \lesssim g$ to mean that for some positive C_1 and C_2, $C_1 \leq f/g \leq C_2$ and $f \leq C_1 g$, respectively. We will also write $f(t) \sim g(t)$, as $t \to \infty$, to mean $f(t)/g(t) \to 1$, as $t \to \infty$.

Lemma 2.1. *Let $0 < \beta \leq 1$. Let ξ be a positive random variable such that for some $c > 0$, $-\log P[\xi > t] \sim ct^\beta$, as $t \to \infty$. Then for independent copies ξ_1 and ξ_2 of ξ, $-\log P[\xi_1 + \xi_2 > t] \sim ct^\beta$, as $t \to \infty$.*

Proof. The lower bound for $P[\xi_1 + \xi_2 > t]$ follows from the observation that $\xi_1 + \xi_2$ is at least ξ_1; so $P[\xi_1 + \xi_2 > t] \geq P[\xi > t]$.

For an upper bound for $P[\xi_1 + \xi_2 > t]$, note that for any $\theta < c$,

$$E(\exp(\theta \xi^\beta)) < \infty.$$

Then by Chebyshev inequality and independence

$$\begin{aligned} P[\xi_1 + \xi_2 > t] &\leq e^{-\theta t^\beta} E(\exp(\theta(\xi_1 + \xi_2)^\beta)) \\ &\leq e^{-\theta t^\beta} E(\exp(\theta(\xi_1^\beta + \xi_2^\beta)) \\ &= e^{-\theta t^\beta} E(\exp(\theta(\xi^\beta))^2, \end{aligned}$$

where we have used the fact that for $0 < \beta \leq 1$ and a, b positive real numbers, $(a+b)^\beta \leq a^\beta + b^\beta$. So,

$$\theta t^\beta - 2 \log E(\exp(\theta(\xi^\beta)) \leq -\log P[\xi_1 + \xi_2 > t] \leq -\log P[\xi > t].$$

Now divide by ct^β, let $t \to \infty$ and $\theta \uparrow c$ to get the desired conclusion. ☐

Lemma 2.2. *Let $\beta > 1$. Let ξ be a positive random variable such that for some $c > 0$, $-\log P[\xi > t] \sim ct^\beta$, as $t \to \infty$. Then*

$$\lim_{t \to \infty} t^{-\beta} \log P[\xi_1 + \xi_2 > t] = -c2^{1-\beta}.$$

Proof. The proof of the lower bound for $P[\xi_1 + \xi_2 > t]$ follows from the fact that

$$P[\xi_1 + \xi_2 > t] \geq (P[\xi > t/2])^2.$$

Indeed, if ξ_1 and ξ_2 are both at least $t/2$, then $\xi_1 + \xi_2 > t$. This implies that for any $\epsilon > 0$, $P[\xi_1 + \xi_2 > t] \geq \exp(-2^{1-\beta} ct^\beta (1 + \epsilon))$ for t large.

For the upper bound for $P[\xi_1 + \xi_2 > t]$ we use the Chebyshev inequality and the fact that for $\beta > 1$ and a, b positive real numbers, $(a+b)^\beta \leq 2^{\beta-1}(a^\beta + b^\beta)$. ☐

We use Lemma 2.1 to derive the asymptotics of the Laplace transform of $(\xi_1+\xi_2)^{-2}$. This is a special case of the following theorem (Kasahara [14, Theorem 3] and Bingham, Goldie and Teugels [3, p. 254].)

Theorem 2.1 (de Bruijn's Tauberian Theorem). *Let X be a positive random variable such that for some positive $B_1,\ B_2$ and p,*

$$-B_1 \le \liminf_{x\to 0} x^p \log P[X\le x] \le \limsup_{x\to 0} x^p \log P[X \le x] \le -B_2.$$

Then

$$\begin{aligned} &-(p+1)(B_1)^{1/(p+1)} p^{-p/(p+1)} \le \liminf_{\lambda\to\infty} \lambda^{-p/(p+1)} \log E e^{-\lambda X} \\ &\le \limsup_{\lambda\to\infty} \lambda^{-p/(p+1)} \log E e^{-\lambda X} \le -(p+1)(B_2)^{1/(p+1)} p^{-p/(p+1)}. \end{aligned}$$

Lemma 2.3. *Let ξ be a positive random variable such that for some $c>0$, $-\log P[\xi > t] \sim ct^\beta$, as $t\to\infty$. Let ξ_1 and ξ_2 be independent copies of ξ. Then, for $0<\beta\le 1$*

$$-\log E(\exp(-\frac{\lambda}{(\xi_1+\xi_2)^2})) \sim ((\beta+2)/2)c^{2/(2+\beta)}(\beta/2)^{-\beta/(\beta+2)}\lambda^{\beta/(\beta+2)},$$

as $\lambda\to\infty$, and for $\beta>1$

$$-\log E(\exp(-\frac{\lambda}{(\xi_1+\xi_2)^2})) \sim ((\beta+2)/2)(c2^{1-\beta})^{2/(\beta+2)}(\beta/2)^{-\beta/(\beta+2)}\lambda^{\beta/(\beta+2)}$$

as $\lambda\to\infty$.

Proof. For $0<\beta\le 1$,

$$-\log P[\frac{1}{(\xi_1+\xi_2)^2} \le x] = -\log P[\xi_1+\xi_2 > x^{-1/2}] \sim cx^{-\beta/2}, \quad \text{as} \quad x\to 0,$$

by Lemma 2.1. For $p=\beta/2$ in de Bruijn's Tauberian Theorem we get

$$-\log E(\exp(-\frac{\lambda}{(\xi_1+\xi_2)^2})) \sim ((\beta+2)/2)c^{2/(2+\beta)}(\beta/2)^{-\beta/(\beta+2)}\lambda^{\beta/(\beta+2)},$$

as $\lambda\to\infty$.

For $\beta>1$, we use Lemma 2.2 and de Bruijn's Tauberian Theorem with, $B_1=B_2=c/2^{\beta-1}$ and $p=\beta/2$. □

We also need the following application of de Bruijn's Tauberian Theorem.

Lemma 2.4. *Let X be a positive random variable with density*

$$f(u) = \gamma u^{-2} e^{-\alpha/u^{\beta/2}},$$

then $-\log\ P[X \leq x] \sim \alpha x^{-\beta/2}$, *as* $x \to 0$. *In this case*

$$-\log\ E(e^{-\lambda X}) \sim ((\beta+2)/2)\alpha^{2/(2+\beta)}(\beta/2)^{-\beta/(\beta+2)}\lambda^{\beta/(\beta+2)},$$

as $\lambda \to \infty$.

Proof.

$$P[X \leq x] = \gamma \int_0^x u^{-2} e^{-\alpha/u^{\beta/2}} du,$$

and making the change of variables $v = u^{-\beta/2}$ gives $u = v^{-2/\beta}$ and $du = -\frac{2}{\beta} v^{-2/\beta - 1} dv$ we get

$$\begin{aligned} P[X \leq x] &= \gamma \frac{2}{\beta} \int_{x^{-\beta/2}}^{\infty} v^{2/\beta - 1} e^{-\alpha v} dv \\ &= \gamma \frac{2}{\beta} \frac{1}{\alpha} (\frac{1}{\alpha})^{1-2/\beta} \int_{\alpha x^{-\beta/2}}^{\infty} z^{2/\beta-1} e^{-z} dz \qquad (2.1) \\ &= \gamma \frac{2}{\beta} \alpha^{-\frac{2}{\beta}} (\alpha x^{-\beta/2})^{2/\beta - 1} e^{-\alpha x^{-\beta/2}} [1 + O(x^{\beta/2})]. \qquad (2.2) \end{aligned}$$

In Equation (2.1) as $x \to 0$, $x^{-\beta/2} \to \infty$. Then equation (2.1) follows by changing variables $\alpha v = z$ and equation (2.2) follows from the asymptotics of incomplete gamma function as in Gradshteyn and Ryzhik [13, p.942]. Hence,

$$x^{\frac{\beta}{2}} \log\ P[X \leq x] = x^{\frac{\beta}{2}}\ \log C_1 + x^{\frac{\beta}{2}} C_2\ \log x - \alpha + x^{\frac{\beta}{2}}\ \log(1 + O(x^{\frac{\beta}{2}}))\ \ \to\ -\alpha,$$

as $x \to 0$, where $C_1 = \gamma \frac{2}{\beta}\frac{1}{\alpha}$ and $C_2 = -\frac{2-\beta}{2}$. Now the desired conclusion follows from de Bruijn's Tauberian Theorem. ☐

3 Proof of main results

If $D \subset \mathbb{R}^n$ is an open set, write

$$\tau_D^{\pm}(z) = \inf\{t \geq 0 :\quad X_t^{\pm} + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, write

$$\eta_I = \eta(I) = \inf\{t \geq 0 : \quad Y_t \notin I\}.$$

Recall that $\tau_D(Z)$ stands for the first exit time of iterated Brownian motion from D. As in DeBlassie [8, §3.], we have by the continuity of the paths for $Z_t = z + X(Y_t)$

$$\begin{aligned} P_z[\tau_D(Z) > t] &= P_z[Z_s \in D \text{ for all } s \leq t] \\ &= P[z + X^+(0 \vee Y_s) \in D \text{ and } z + X^-(0 \vee (-Y_s)) \in D \\ &\quad \text{for all } s \leq t] \\ &= P[\tau_D^+(z) > 0 \vee Y_s \text{ and } \tau_D^-(z) > 0 \vee (-Y_s) \text{ for all } s \leq t] \\ &= P[-\tau_D^-(z) < Y_s < \tau_D^+(z) \text{ for all } s \leq t] \\ &= P[\eta(-\tau_D^-(z), \tau_D^+(z)) > t], \end{aligned} \tag{3.1}$$

and this equals for the parabola-shaped domains and twisted domains introduced above

$$= \int_0^\infty \int_0^\infty \left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} P_0[\eta_{(-u,v)} > t] \right) P[\tau_D(z) > u] P[\tau_D(z) > v] dv du. \tag{3.2}$$

The equation (3.2) follows from Lemma 4.4 below.

For the parabola-shaped domains $P_\alpha \subset \mathbb{R}^n$ we have

$$\lim_{t \to \infty} t^{-(\frac{1-\alpha}{1+\alpha})} \log P_z[\tau_\alpha > t] = -l,$$

where l is given in (1.2). Similarly for the twisted domains $D \subset \mathbb{R}^2$

$$\lim_{t \to \infty} t^{-(\frac{1-p}{1+p})} \log P_z[\tau_D > t] = -l_1,$$

where l_1 is given in (1.3). Then our main theorem, Theorem 1.1, follows from equation (3.1) and by substituting $(1-\alpha)/(1+\alpha)$ for β and l for c in the following theorem. Similarly Theorem 1.2 follows from the following theorem as well from equation (3.1) and by substituting $(1-p)/(1+p)$ for β and l_1 for c.

The following theorem, Theorem 3.1, is more general than DeBlassie's theorem [8, Theorem 4.4], we use the asymptotics of the distribution of the random variables rather than the density of the random variables. For this, we use integration by parts and the asymptotics of $\frac{\partial}{\partial u} \frac{\partial}{\partial v} P_0[\eta_{(-u,v)} > t]$.

Theorem 3.1. *Let $0 < \beta \leq 1$. Let ξ be a positive random variable such that $-\log P[\xi > t] \sim ct^\beta$, as $t \to \infty$. If ξ_1 and ξ_2 are independent copies of ξ and independent of the Brownian motion Y, then*

$$-\log P[\eta_{(-\xi_1,\xi_2)} > t] \sim (\frac{2+\beta}{2}) c^{2/(2+\beta)} \beta^{-\beta/(2+\beta)} \pi^{2\beta/(2+\beta)} t^{\beta/(2+\beta)},$$

as $t \to \infty$.

Proof. We suppose $Y_0 = x$ and the probability associated with this will be P_x. The distribution of $\eta_{(-u,v)}$ is well known:

$$P_0[\eta_{(-u,v)} > t] = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2} t) \sin \frac{(2n+1)\pi u}{u+v}, \quad (3.3)$$

see Feller [11, pp. 340-342].
Let $\epsilon > 0$. From Lemma 4.1, choose $M > 0$ so large that

$$P_x[\eta_{(0,1)} > t] \approx \frac{4}{\pi} e^{-\frac{\pi^2 t}{2}} \sin \pi x, \text{ for } t \geq M \text{ uniformly } \ x \in (0,1). \quad (3.4)$$

Then choose $\delta < \frac{1}{2}$ so small that

$$\sin \pi x \approx x, \quad x \in (0, \delta]. \quad (3.5)$$

From the hypothesis choose $K > 0$ so large that

$$e^{-c(1+\epsilon)u^\beta} \leq P(\xi > u) \leq e^{-c(1-\epsilon)u^\beta} \text{ for } \ u \geq K. \quad (3.6)$$

We further assume that t is so large that $K < \delta\sqrt{t/M}$.

If f is the probability density function of the random variable ξ, by independence of Y, ξ_1 and ξ_2, and using scaling and translation invariance of Brownian motion

$$\begin{aligned} P[\eta_{(-\xi_1,\xi_2)} > t] &= P_0[\eta_{(-\xi_1,\xi_2)} > t] \\ &= \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) dv du \quad (3.7) \\ &= \int_0^\infty \int_0^\infty P_{\frac{u}{(u+v)}}[\eta_{(0,1)} > \frac{t}{(u+v)^2}] f(u) f(v) dv du, (3.8) \end{aligned}$$

and this equals

$$= \int_0^\infty \int_0^\infty \left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} P_0[\eta_{(-u,v)} > t] \right) P[\xi > u] P[\xi > v] dv du. \quad (3.9)$$

Equation (3.9) follows from Lemma 4.4 using integration by parts in (3.7). By Lemma 4.3 the integral in equation (3.9) over the set $u+v \geq \sqrt{t/M}$ satisfies

$$\int\int_{u+v\geq\sqrt{t/M}} \left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]\right) P[\xi > u]P[\xi > v]dvdu$$
$$\gtrsim -e^{-C_0(\sqrt{t/M})^\beta}. \qquad (3.10)$$

For large t, let

$$A = \left\{(u,v): \quad K \leq u \leq \delta\sqrt{\frac{t}{M}},\ \frac{1-\delta}{\delta}u \leq v \leq \sqrt{\frac{t}{M}} - u\right\}.$$

For $u+v \leq \sqrt{t/M}$ we have the lower bound approximation from Lemma 4.2 which says for $u+v \leq \sqrt{t/M}$

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t] \gtrsim \ \exp(-\frac{\pi^2 t}{2(u+v)^2})\frac{uv}{(u+v)^4}\sin\frac{\pi u}{u+v}. \qquad (3.11)$$

Now $A \subset \{(u,v) : u+v \leq \sqrt{t/M}\}$. Then by equations (3.9)-(3.11) we get

$$\begin{aligned} P[\eta_{(-\xi_1,\xi_2)} > t] \ &\gtrsim \ \int\int_A \left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]\right) P[\xi > u]P[\xi > v]dvdu \\ &+ \ -e^{-C_0(\sqrt{t/M})^\beta}. \end{aligned}$$

We will show below that the integral on the right hand side of the last inequality is $\gtrsim \exp(-C_1 t^{\beta/(\beta+2)})$ for some C_1 positive. Thus, using the fact that for t large

$$\exp(-C_1 t^{\beta/(\beta+2)}) - \exp(-C_0(\sqrt{t/M})^\beta) \gtrsim \exp(-C_1 t^{\beta/(\beta+2)}),$$

in the rest of the proof we will omit the second term, $\exp(-C_0(\sqrt{t/M})^\beta)$, in the above inequality in finding the lower bound estimate for $P[\eta_{(-\xi_1,\xi_2)} > t]$.

On the set A, since $\delta < 1/2$, we have $v \geq (\frac{1}{\delta}-1)u > u > K$ and $u+v > \frac{u}{\delta}$, this gives $\frac{u}{u+v} \leq \delta$. By equations (3.5), (3.6) and (3.11), $P[\eta_{(-\xi_1,\xi_2)} > t]$ is

$$\gtrsim \int_K^{\delta\sqrt{t/M}} \int_{(1-\delta)u/\delta}^{\sqrt{t/M}-u} \exp(-\frac{\pi^2 t}{2(u+v)^2})\frac{u^2 v}{(u+v)^5}e^{-(1+\epsilon)c(u^\beta+v^\beta)}dvdu,$$

and this is

$$\gtrsim \int_K^{\delta\sqrt{t/M}} \int_{(1-\delta)u/\delta}^{\sqrt{t/M}-u} \exp(-\frac{\pi^2 t}{2(u+v)^2})\frac{u}{(u+v)^5} e^{-(1+\epsilon)c(1+\delta^\beta)(u+v)^\beta} dvdu. \tag{3.12}$$

Inequality (3.12) follows from the fact that $u < \delta(u+v)$ over the set A which gives $u^\beta + v^\beta \leq (1+\delta^\beta)(u+v)^\beta$, and $uv \geq K^2$ over A. Changing the variables $x = u+v, z = u$ the integral is

$$\approx \int_K^{\delta\sqrt{t/M}} \int_{z/\delta}^{\sqrt{t/M}} \exp(-\frac{\pi^2 t}{2x^2})\frac{z}{x^5} e^{-(1+\epsilon)c(1+\delta^\beta)x^\beta} dxdz,$$

and reversing the order of integration

$$\begin{aligned}
&= \int_{K/\delta}^{\sqrt{t/M}} \int_K^{\delta x} \frac{z}{x^5} \exp(-\frac{\pi^2 t}{2x^2}) e^{-(1+\epsilon)c(1+\delta^\beta)x^\beta} dzdx \\
&\approx \int_{K/\delta}^{\sqrt{t/M}} \frac{1}{x^5} \exp(-\frac{\pi^2 t}{2x^2}) e^{-(1+\epsilon)c(1+\delta^\beta)x^\beta}\ (\delta^2 x^2 - K^2) dx \\
&\geq \int_{2K/\delta}^{\sqrt{t/M}} \frac{1}{x^5} \exp(-\frac{\pi^2 t}{2x^2}) e^{-(1+\epsilon)c(1+\delta^\beta)x^\beta}\ (\delta^2 x^2 - K^2) dx, \quad t \text{ large} \\
&\geq \int_{2K/\delta}^{\sqrt{t/M}} \frac{1}{x^5} \exp(-\frac{\pi^2 t}{2x^2}) e^{-(1+\epsilon)c(1+\delta^\beta)x^\beta}\ (\delta^2 x^2 - \frac{\delta^2 x^2}{4}) dx.
\end{aligned}$$

Changing variables $u = x^{-2}$ this is

$$\begin{aligned}
&\approx \int_{M/t}^{\delta^2/4K^2} \exp(-\pi^2 tu/2) e^{-(1+\epsilon)c(1+\delta^\beta)(u)^{-\beta/2}}\ du \\
&\geq (M/t)^2 \int_{M/t}^{\delta^2/4K^2} u^{-2} \exp(-\pi^2 tu/2) e^{-(1+\epsilon)c(1+\delta^\beta)(u)^{-\beta/2}} du.
\end{aligned}$$

Thus we have for t large,

$$\begin{aligned}
&P[\eta_{(-\xi_1,\xi_2)} > t] \\
&\gtrsim (M/t)^2 \int_{M/t}^{\delta^2/4K^2} u^{-2} \exp(-\pi^2 tu/2) e^{-(1+\epsilon)c(1+\delta^\beta)(u)^{-\beta/2}}\ du.
\end{aligned} \tag{3.13}$$

We can disregard $(M/t)^2$ since we will take log and divide by $t^{\beta/(2+\beta)}$ and let $t \to \infty$. That is, $t^{-\beta/(\beta+2)} \log t \to 0$, as $t \to \infty$. Now for some $c_1 > 0$

$$\begin{aligned}
&\int_{\delta^2/4K^2}^{\infty} u^{-2} \exp(-\pi^2 tu/2) e^{-(1+\epsilon)c(1+\delta^\beta)u^{-\beta/2}}\ du \\
&\le e^{-\pi^2\delta^2 t/8K^2} \int_{\delta^2/4K^2}^{\infty} u^{-2} e^{-(1+\epsilon)c(1+\delta^\beta)u^{-\beta/2}}\ du \\
&\lesssim e^{-c_1 t}. \qquad (3.14)
\end{aligned}$$

Changing variables $u = v^{-2}$

$$\begin{aligned}
&\int_0^{M/t} u^{-2} \exp(-\pi^2 tu/2) e^{-(1+\epsilon)c(1+\delta^\beta)u^{-\beta/2}}\ du \\
&\le \int_0^{M/t} u^{-2} e^{-(1+\epsilon)c(1+\delta^\beta)u^{-\beta/2}}\ du \\
&= 2\int_{\sqrt{t/M}}^{\infty} v e^{-(1+\epsilon)c(1+\delta^\beta)v^\beta} dv \\
&\lesssim (t/M)^{\beta/2(2/\beta-1)} e^{-(1+\epsilon)c(1+\delta^\beta)(t/M)^{\beta/2}}. \qquad (3.15)
\end{aligned}$$

The inequality (3.15) follows from the asymptotics of the incomplete gamma function as in Gradshteyn and Ryzhik [13, p.942]. By Lemma 2.4, with $\alpha = c(1+\epsilon)(1+\delta^\beta)$ and $\lambda = \frac{\pi^2 t}{2}$

$$\begin{aligned}
&-\log \int_0^{\infty} u^{-2} \exp(-\pi^2 tu/2) e^{-(1+\epsilon)c(1+\delta^\beta)u^{-\beta/2}}\ du \\
&\sim (\frac{\beta}{2}+1)(c(1+\epsilon)(1+\delta^\beta))^{1/(\frac{\beta}{2}+1)} (\frac{\beta}{2})^{-\frac{\beta}{2}/(\frac{\beta}{2}+1)} (\frac{\pi^2 t}{2})^{\frac{\beta}{2}/(\frac{\beta}{2}+1)} \\
&= (\frac{2+\beta}{2})(c(1+\epsilon)(1+\delta^\beta))^{\frac{2}{(2+\beta)}} \beta^{-\frac{\beta}{(2+\beta)}} \pi^{\frac{2\beta}{(2+\beta)}} t^{\frac{\beta}{(2+\beta)}}, \qquad (3.16)
\end{aligned}$$

as $t \to \infty$.

By equations (3.10), (3.14)- (3.16) and from the fact that $\beta/(\beta+2) < \beta/2$ and $\beta/(\beta+2) < 1$, we can rewrite equation (3.13) for large t as $P[\eta_{(-\xi_1,\xi_2)} > t]$ is

$$\gtrsim \exp\left(-(1+\epsilon)(\frac{2+\beta}{2})(c(1+\epsilon)(1+\delta^\beta))^{\frac{2}{(2+\beta)}} \beta^{-\frac{\beta}{(2+\beta)}} \pi^{\frac{2\beta}{(2+\beta)}} t^{\frac{\beta}{(2+\beta)}}\right). \qquad (3.17)$$

Now we give an upper bound. By equations (3.4) and (3.8)

$$
\begin{aligned}
P[\eta_{(-\xi_1,\xi_2)} > t] &= \int_0^\infty \int_0^\infty P_{u/(u+v)}[\eta_{(0,1)} > \frac{t}{(u+v)^2}]\ f(u)f(v)dvdu \\
&\lesssim \iint_{u+v\leq\sqrt{t/M}} e^{-\frac{\pi^2 t}{2(u+v)^2}}\ f(u)f(v)dvdu \\
&+ \iint_{u+v\geq\sqrt{t/M}} \quad f(u)f(v)dvdu \\
&\leq E\left[\exp(-\frac{\pi^2 t}{2(\xi_1+\xi_2)^2})\right] + P(\xi_1+\xi_2 \geq \sqrt{t/M}). \quad (3.18)
\end{aligned}
$$

By Lemma 2.1 our random variables ξ_1 and ξ_2 satisfy

$$
-\log P[\xi_1+\xi_2 > t] \sim ct^\beta, \quad \text{as} \quad t \to \infty.
$$

Then by Lemma 2.3 for t large, equation (3.18) becomes

$$
\begin{aligned}
&P[\eta_{(-\xi_1,\xi_2)} > t] \\
&\lesssim \exp\left(-(1-\epsilon)(\frac{2+\beta}{2})c^{\frac{2}{(2+\beta)}}\beta^{-\frac{\beta}{(2+\beta)}}\pi^{\frac{2\beta}{(2+\beta)}}t^{\frac{\beta}{(2+\beta)}}\right) \\
&+\ \exp\left(-c(1-\epsilon)(t/M)^{\beta/2}\right) \\
&\lesssim \exp\left(-(1-\epsilon)(\frac{2+\beta}{2})c^{\frac{2}{(2+\beta)}}\beta^{-\frac{\beta}{(2+\beta)}}\pi^{\frac{2\beta}{(2+\beta)}}t^{\frac{\beta}{(2+\beta)}}\right).
\end{aligned}
$$

Combined with inequality (3.17), this gives

$$
\begin{aligned}
&-(1+\epsilon)(\frac{2+\beta}{2})(c(1+\epsilon)(1+\delta^\beta))^{\frac{2}{(2+\beta)}}\beta^{-\frac{\beta}{(2+\beta)}}\pi^{\frac{2\beta}{(2+\beta)}} \\
&\leq \liminf_{t\to\infty}\ t^{-\beta/(2+\beta)}\ \log\ P[\eta_{(-\xi_1,\xi_2)} > t] \\
&\leq \limsup_{t\to\infty}\ t^{-\beta/(2+\beta)}\ \log\ P[\eta_{(-\xi_1,\xi_2)} > t] \\
&\leq -(1-\epsilon)(\frac{2+\beta}{2})c^{\frac{2}{(2+\beta)}}\beta^{-\frac{\beta}{(2+\beta)}}\pi^{\frac{2\beta}{(2+\beta)}}.
\end{aligned}
$$

Let $\epsilon \to 0$ and $\delta \to 0$ to get the desired conclusion. $\square$

As a corollary of Theorem 3.1 we obtain DeBlassie's result [8, Theorem 4.4].

Corollary 3.1. *Let $0 < \beta \leq 1$. Let ξ be a positive random variable with a density function f such that $-\log f(t) \sim ct^{\beta}$, as $t \to \infty$. If ξ_1 and ξ_2 are independent copies of ξ and independent of the Brownian motion Y, then*

$$-\log P[\eta_{(-\xi_1,\xi_2)} > t] \sim (\frac{2+\beta}{2})c^{2/(2+\beta)}\beta^{-\beta/(2+\beta)}\pi^{2\beta/(2+\beta)}t^{\beta/(2+\beta)},$$

as $t \to \infty$.

The proof is immediate after we observe that if $-\log f(t) \sim ct^{\beta}$, as $t \to \infty$, then

$$-\log P[\xi > t] \sim ct^{\beta}, \text{ as } t \to \infty.$$

We have the following corollary of Theorem 3.1.

Theorem 3.2. *Let $0 < \beta \leq 1$. Let D be a domain in $\mathbb{R}^n$. Let τ_D denote the first exit time of the Brownian motion from D and satisfy for $z \in D$, $\lim_{t\to\infty} t^{-\beta} \log P_z[\tau_D > t] = -c$ for some c positive. If $\tau_D(Z)$ denotes the first exit time of the iterated Brownian motion from D, then*

$$\lim_{t\to\infty} t^{-\beta/(2+\beta)} \log P[\tau_D(Z) > t] = -(\frac{2+\beta}{2})c^{2/(2+\beta)}\beta^{-\beta/(2+\beta)}\pi^{2\beta/(2+\beta)}.$$

Actually, Theorem 1.2 follows from Theorem 3.2 above by substituting $(1-p)/(1+p)$ for β and for c the limit in equation (1.3). Theorem 1.1 as well follows from Theorem 3.2 in a similar way.

Theorem 3.3. *Let $\beta > 1$. Let ξ be a positive random variable such that $-\log P[\xi > t] \sim ct^{\beta}$, as $t \to \infty$. If ξ_1 and ξ_2 are independent copies of ξ and independent of the Brownian motion Y, then*

$$\begin{aligned}
&-(\frac{2+\beta}{2})c^{\frac{2}{(2+\beta)}}\beta^{-\frac{\beta}{(2+\beta)}}\pi^{\frac{2\beta}{(2+\beta)}} \\
&\leq \liminf_{t\to\infty}\ t^{-\beta/(2+\beta)}\ \log\ P[\eta_{(-\xi_1,\xi_2)} > t] \\
&\leq \limsup_{t\to\infty}\ t^{-\beta/(2+\beta)}\ \log\ P[\eta_{(-\xi_1,\xi_2)} > t] \\
&\leq -(\frac{2+\beta}{2})(c2^{1-\beta})^{\frac{2}{(2+\beta)}}\beta^{-\frac{\beta}{(2+\beta)}}\pi^{\frac{2\beta}{(2+\beta)}}.
\end{aligned}$$

Proof. The result follows from Lemma 2.2 , de Bruijn's Tauberian Theorem and the proof of the Theorem 3.1, using the well-known fact that for $\beta > 1$ and $a,\ b$ positive real numbers, $(a+b)^{\beta} \leq 2^{\beta-1}(a^{\beta}+b^{\beta})$. □

Comparing with a ball inside any domain D, we see that the analogue of the Theorem 3.2 does not hold for $\beta > 1$. Let $B \subset D$ be a ball centered at x, then for the Brownian motion X_t started at $x \in D$, $-\log P_x[\tau_D > t] \lesssim \lambda_B t$ for large t where λ_B is the first eigenvalue of the Dirichlet Laplacian for B. This implies that the statement ' $-\log P_x[\tau_D > t] \sim ct^\beta$, as $t \to \infty$ ' cannot be true for $\beta > 1$.

4 Asymptotics

In this Section we will prove some lemmas that were used in section 3. The following lemma is proved in [8, Lemma A1] (it also follows from more general results on "intrinsic ultracontractivity"). We include it for completeness.

Lemma 4.1. *As $t \to \infty$,*

$$P_x[\eta_{(0,1)} > t] \sim \frac{4}{\pi} e^{-\frac{\pi^2 t}{2}} \sin \pi x, \quad \text{uniformly for} \quad x \in (0,1).$$

We will need asymptotics of $\frac{\partial}{\partial u}\frac{\partial}{\partial v} P_0(\eta_{(-u,v)} > t)$ for $(u,v) \in A$, where we define A for $\delta < 1/2$, $K > 0$ and $M > 0$ as

$$A = \left\{ (u,v) : \quad K \le u \le \delta \sqrt{\frac{t}{M}}, \ \frac{1-\delta}{\delta} u \le v \le \sqrt{\frac{t}{M}} - u \right\}.$$

Lemma 4.2. *Let $B = \{(u,v) : \ t/(u+v)^2 > M\}$ for M large. On B we have*

$$\begin{aligned}
&\frac{\partial}{\partial u}\frac{\partial}{\partial v} P[\eta_{(-u,v)} > t] \\
\approx \quad & 4 \exp(-\frac{\pi^2 t}{2(u+v)^2}) \Bigg((\sin \frac{\pi u}{u+v})(\frac{1}{(u+v)^4})(\frac{\pi^3 t^2}{(u+v)^2} - 3\pi t \\
&+ \pi uv) + (\cos \frac{\pi u}{u+v})(\frac{1}{(u+v)^3})(\frac{\pi^2 t(v-u)}{(u+v)^2} + u - v) \Bigg).
\end{aligned}$$

Moreover, on B

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v} P_0[\eta_{(-u,v)} > t] \gtrsim \exp(-\frac{\pi^2 t}{2(u+v)^2}) \frac{uv}{(u+v)^4} \sin \frac{\pi u}{u+v}. \tag{4.1}$$

Proof. If we take the derivative of

$$P_0[\eta_{(-u,v)} > t] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t)\ \sin\ \frac{(2n+1)\pi u}{u+v},$$

term by term w.r.t u, we get

$$\begin{aligned}
&\frac{\partial}{\partial u}P_0[\eta_{(-u,v)} > t] \qquad\qquad (4.2)\\
&= \frac{4}{\pi}\sum_{n=0}^{\infty}\ \exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t)\left[\frac{(2n+1)\pi^2 t}{(u+v)^3}\ \sin\ \frac{(2n+1)\pi u}{u+v}\right.\\
&\qquad\qquad \left.+\frac{\pi v}{(u+v)^2}\ \cos\ \frac{(2n+1)\pi u}{u+v}\right],
\end{aligned}$$

and if we take derivative of equation (4.2) with respect to v, then

$$\begin{aligned}
&\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]\\
&= 4\ \sum_{n=0}^{\infty}\ \exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t)\left\{\left[\ \sin\ \frac{(2n+1)\pi u}{u+v}\ (\frac{1}{(u+v)^4})\right.\right.\\
&\qquad \left.\times\ (\frac{\pi^3(2n+1)^3t^2}{(u+v)^2} -\ 3\pi(2n+1)t\ +\ (2n+1)\pi uv)\ \right]\\
&+\left.\left[\ \cos\ \frac{(2n+1)\pi u}{u+v}\ (\frac{1}{(u+v)^3})(\frac{(2n+1)^2\pi^2 t(v-u)}{(u+v)^2} +\ \ u\ -\ v)\right]\right\}.
\end{aligned}$$

The term by term differentiation is admissible since we have the exponential function in each term.

The asymptotics in the lemma related to sin terms follow from the proof of Lemma 4.1 in Deblassie [8, Lemma A1], which says that for some $c > 0$

$$|\sin(2n+1)\pi x/\sin\pi x| \le c(2n+1)^2,\ \text{uniformly for all } x\in(0,1).$$

The asymptotics for the cosine terms follow from the fact that by induction on n,

$$|\cos(2n+1)\pi x/\cos\pi x| \le 2n+1,\ \text{uniformly for all } x\in(0,1).$$

The lower bound asymptotics follow from the fact that for $u+v < \sqrt{t/M}$

$$\frac{\pi^3 t^2}{(u+v)^2} - 3\pi t \ge 0,$$

$$(\frac{\pi^2 t}{(u+v)^2} - 1)) > 0,$$

and

$$(v-u)\cos\frac{\pi u}{u+v} \geq 0 \ \text{ for } \ \text{all} \ \ u, v > 0.$$

The last statement is valid since for $u \leq v$ we have $(v-u) \geq 0$ and $u/(u+v) \leq 1/2$ which gives $\cos\frac{\pi u}{u+v} \geq 0$, hence their product is positive. For $u \geq v$ we have $v - u \leq 0$ and $\cos\frac{\pi u}{u+v} \leq 0$, so their product is positive. Hence we get the lower bound asymptotics for $t/(u+v)^2$ large. □

Note that we obtain lower bound asymptotics of $\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]$ on the set A, because $A \subset B$.

Lemma 4.3. *Let $\beta > 0$. Let ξ be a positive random variable such that $-\log P[\xi > t] \sim ct^\beta$, as $t \to \infty$. If ξ_1 and ξ_2 are independent copies of ξ and independent of the Brownian motion Y, then for some $C_0 > 0$*

$$\int\int_{u+v \geq \sqrt{t/M}} \left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]\right) P[\xi > u]P[\xi > v]dvdu$$
$$\gtrsim -e^{-C_0(\sqrt{t/M})^\beta}.$$

Proof. We divide the set $u + v \geq \sqrt{t/M}$ into the following subsets

$$A_k = \{(u,v) : k\sqrt{t/M} \leq u + v < (k+1)\sqrt{t/M})\}.$$

Now

$$\begin{aligned}
&\int\int_{u+v \geq \sqrt{t/M}} \left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]\right) \ P[\xi > u]P[\xi > v]dvdu \\
&= \sum_{k=1}^{\infty}\int\int_{A_k} \left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}P_0[\eta_{(-u,v)} > t]\right) \ P[\xi > u]P[\xi > v]dvdu \\
&\gtrsim -\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\int\int_{A_k}(U+V)P[\xi > u]P[\xi > v]dvdu, \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
U \ &= \ \exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t) \ (\frac{1}{(u+v)^4}) \\
&\quad \times \left|(\frac{\pi^3(2n+1)^3t^2}{(u+v)^2} \ - \ 3\pi(2n+1)t + \ (2n+1)\pi uv)\right|,
\end{aligned}$$

and

$$V = \exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t) \left| (\frac{1}{(u+v)^3})(u - v + \frac{(2n+1)^2\pi^2 t(v-u)}{(u+v)^2}) \right|.$$

We work on each term in equation (4.3). On the set A_k,

$$\exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t) \leq e^{-((2n+1)/(k+1))^2 M/2},$$

and terms like the reciprocal powers of $u+v$ are less than the reciprocal of k with the same power. Hence we have from equation (4.3) for t large

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\int\int_{A_k}(U+V)P[\xi > u]P[\xi > v]dvdu \\
&\lesssim \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\int\int_{A_k} e^{-((2n+1)/(k+1))^2 M/2} \\
&\times \left[\left((\frac{1}{k^4})(\frac{\pi^3(2n+1)^3 t^2}{k^2} + 3\pi(2n+1)t) + ((2n+1)\pi\frac{1}{k^2}) \right) \right. \\
&\quad \left. + \left(\frac{1}{(k)^2}(\frac{(2n+1)^2\pi^2 t}{k^2} \ + \ 1) \right) \right] P[\xi > u]P[\xi > v]dvdu. \quad (4.4)
\end{aligned}$$

Now $A_k \subset A_k^1 \cup A_k^2$, where

$$A_k^1 = \{(u,v) : k/2\sqrt{t/M} \leq u \leq (k+1)\sqrt{t/M} \text{ and } 0 \leq v \leq (k+1)/2\sqrt{t/M}\}$$

$$A_k^2 = \{(u,v) : k/2\sqrt{t/M} \leq v \leq (k+1)\sqrt{t/M} \text{ and } 0 \leq u \leq (k+1)/2\sqrt{t/M}\}.$$

From the asymptotics of $P[\xi > t]$, for t large there is some $C_0 > 0$ independent of k such that $P[\xi > u]P[\xi > v] \leq \exp(-C_0 u^\beta)$, for $(u,v) \in A_k^1$. Thus by symmetry, same holds for $(u,v) \in A_k^2$

$$\begin{aligned}
&\int\int_{A_k} P[\xi > u]P[\xi > v] \leq \int\int_{A_k^1 \cup A_k^2} P[\xi > u]P[\xi > v] \\
&\lesssim (k+1)/2\sqrt{t/M}\int_{k/2\sqrt{t/M}}^{(k+1)\sqrt{t/M}} e^{-C_0 u^\beta} du \\
&\lesssim (k+1)/2\sqrt{t/M}(k+2)/2\sqrt{t/M} e^{-C_0(\frac{k}{2}\sqrt{t/M})^\beta},
\end{aligned}$$

(4.4) becomes

$$\leq \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(k+1)/2\sqrt{t/M}(k+2)/2\sqrt{t/M}e^{-C_0(\frac{k}{2}\sqrt{t/M})^\beta}$$
$$\times\ e^{-\frac{(2n+1)^2}{(k+1)^2}M/2}\left[\frac{1}{k^4}\left(\frac{\pi^3(2n+1)^3t^2}{k^2}+3\pi(2n+1)t\right)+\frac{(2n+1)\pi}{k^2}\right.$$
$$\left.+\left(\frac{1}{k^2}(\frac{(2n+1)^2\pi^2 t}{k^2}\ +\ 1)\right)\right] \tag{4.5}$$

We work on each sum in (4.5) in n separately, but they are all similar. We first consider the second term in the square brackets in (4.5). Now for $|x|<1$,

$$\sum_{n=0}^{\infty} n x^{n^2} \leq \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

Thus

$$\sum_{n=0}^{\infty} e^{-((2n+1)/(k+1))^2M/2}((2n+1)\pi) \leq \pi\frac{e^{-(1/(k+1))^2M/2}}{(1-e^{-(1/(k+1))^2M/2})^2}$$
$$\leq \pi\frac{1}{(1-e^{-(1/(k+1))^2M/2})^2},$$

Now since $C_0(\sqrt{t/M}1/2)^\beta > 1$ for t large,

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{(k+1)}{2}(k+2)/2(\sqrt{t/M})^2\frac{e^{-C_0(\frac{k}{2}\sqrt{t/M})^\beta}}{(1-e^{-(1/(k+1))^2M/2})^2}$$
$$\leq (t/M)e^{-C_0(\frac{1}{2}\sqrt{t/M})^\beta}$$
$$\times\ \sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{4k^2}\frac{e^{-(k^\beta-1)}}{(1-e^{-(1/(k+1))^2M/2})^2}.$$

Now since

$$(k+1)(k+2)\frac{e^{-(k^\beta-1)}}{(1-e^{-(1/(k+1))^2M/2})^2}$$

is bounded uniformly for all k, actually this quantity tends to zero as k tends to infinity, we get the desired conclusion. Alternately, for some $p>0$ and $C>0$ independent of k we have

$$(k+1)(k+2)\frac{e^{-(k^\beta-1)}}{(1-e^{-(1/(k+1))^2M/2})^2} \leq k^p e^{-Ck^\beta}.$$

Then since $\sum_{k=1}^{\infty} k^p e^{-Ck^\beta} < \infty$, we get the desired lower bound. All other terms are handled in a similar fashion; we use the following power series expansions for $|x| < 1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)},$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3},$$

and

$$\sum_{n=0}^{\infty} n^3 x^n = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

□

Lemma 4.4. *Let $\beta > 0$. Let ξ be a positive random variable which satisfies $-\log P[\xi > t] \sim ct^\beta$, as $t \to \infty$. Let ξ_1 and ξ_2 be independent copies of ξ and independent of the Brownian motion Y. Then*

$$\begin{aligned} P[\eta_{(-\xi_1,\xi_2)} > t] &= P_0[\eta_{(-\xi_1,\xi_2)} > t] \\ &= \int_0^\infty \int_0^\infty \left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} P_0(\eta_{(-u,v)} > t) \right) P[\xi > u] P[\xi > v] dv du. \end{aligned}$$

Proof. The proof follows from integration by parts in equation (3.7), if we can show the following two statements

$$\lim_{u \to 0} P_0[\eta_{(-u,v)} > t] P[\xi > u] = \lim_{u \to \infty} P_0[\eta_{(-u,v)} > t] P[\xi > u] = 0, \tag{4.6}$$

$$\lim_{v \to 0} \left(\frac{\partial}{\partial u} P_0[\eta_{(-u,v)} > t]\right) P[\xi > v] = \lim_{v \to \infty} \left(\frac{\partial}{\partial u} P_0[\eta_{(-u,v)} > t]\right) P[\xi > v] = 0. \tag{4.7}$$

The equation (4.6) follows from the formula for $P_0[\eta_{(-u,v)} > t]$ above in equation (3.3) and the equation (4.7) is obvious from the asymptotics for $P[\xi > u]$ and equation (4.2). □

Acknowledgments. I would like to thank Professor Rodrigo Bañuelos, my academic advisor, for suggesting me this problem and for his guidance on this paper. I also would like to thank Professor Dante DeBlassie and Professor Davar Khoshnevisan for useful comments on the early drafts of this work.

References

[1] Bañuelos, R., DeBlassie, R.D., Smits, R.: 'The first exit time of planar Brownian motion from the interior of a parabola', *Ann. Prob.* (2) **29** (2001), 882 - 901.

[2] van den Berg, M.: 'Subexponential behavior of the Dirichlet heat kernel', *J. Funct. Anal.* **198** (2003), 28 - 42.

[3] Bingham, N.H., Goldie, C.M., Teugels, J.L.: *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[4] Burdzy, K.: 'Some path properties of iterated Brownian motion', In Seminar on Stochastic Processes (E. Çinlar, K.L. Chung and M.J. Sharpe, eds.), *Birkhäuser*, Boston, (1993), 67 - 87.

[5] Burdzy, K.: 'Variation of iterated Brownian motion', In Workshops and Conference on Measure-valued Processes, Stochastic Partial Differential Equations and Interacting Particle Systems (D.A. Dawson, ed.) *Amer. Math. Soc.* Providence, RI, (1994),35 - 53.

[6] Burdzy, K., Khoshnevisan, D.: 'Brownian motion in a Brownian crack', *Ann. Appl. Probabl.* (3) **8** (1998), 708 - 748.

[7] Csàki, E., Csörgő, M., Földes, A., Révész, P.: 'The local time of iterated Brownian motion', *J. Theoret. Probab.* (3) **9** (1996), 717 - 743.

[8] DeBlassie, R. D.: 'Iterated Brownian motion in an open set', Preprint, 2004.

[9] DeBlassie, R.D., Smits, R.: 'Brownian motion in twisted domains', Preprint, 2004.

[10] Eisenbum, N., Shi, Z.: 'Uniform oscillations of the local time of iterated Brownian motion', *Bernoulli* (1) **5** (1999), 49 - 65.

[11] Feller, W.: *An Introduction to Probability Theory and its Applications*, Wiley, New York, 1971.

[12] Funaki, T.: 'A probabilistic construction of the solution of some higher order parabolic differential equations', *Proc. Japan Acad. Ser. A. Math. Sci.* (5) **55** (1979), 176 - 179.

[13] Gradshteyn, I. S., Ryzhik, I. M.: *Table of Integrals, Series and Products*, Academic, New York, 1965.

[14] Kasahara, Y.: 'Tauberian theorems of exponential type', *J. Math. Kyoto Univ.* (2) **12** (1978), 209 - 219.

[15] Khoshnevisan, D., Lewis, T.M.: 'Stochastic calculus for Brownian motion in a Brownian fracture', *Ann. Applied Probabl.* (3) **9** (1999), 629 - 667.

[16] Lifshits, M., Shi, Z.: 'The first exit time of Brownian motion from a parabolic domain', *Bernoulli* (6) **8** (2002), 745 - 765.

[17] Xiao, Y.: 'Local times and related properties of multidimensional iterated Brownian motion', *J. Theoret. Probab.* (2) **11** (1998), 383 - 408.